B.Sc Part I (Physics Hons)
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Question:- What is Doppler's effect. Discuss relativistically the phenomenon of Doppler effect in light.

Ans **Doppler Effect** :→

An apparent change in frequency of received radiation due to relative motion between source, observer and medium is known Doppler effect.

Let us consider a plane wave whose wave ~~motion~~ normal points in direction given by direction cosines l, m and n w.r.to an observer in the frame S. Let its phase velocity ~~be~~ be 'u' such that its components are ul, um & un. Let its wavelength, frequency and Time period be λ, ν and τ respectively such that $\lambda = \frac{u}{\nu} = u\tau$. These quantities in frame S' are written with primed symbols. Let us take an equation of plane wave as

$$\psi = A \exp\left[2\pi i\left(\frac{lx+my+nz}{\lambda} - \frac{t}{\tau}\right)\right] \quad \text{------} (1)$$

$$= A \exp\left[\vec{K}\cdot\vec{x} - \frac{2\pi i t}{\tau}\right] \quad \text{------} (2)$$

Here vector $\vec{K} = \frac{2\pi}{\lambda}(l, m, n)$ and points in the direction of wave propagation and $\vec{x} = (x, y, z)$

Now Let us consider a set of four quantities

$$(K_i) = \left(\vec{K} \cdot \frac{i\omega}{c}\right), \quad \because \omega = \frac{2\pi}{\tau} = 2\pi\nu$$

from eqn (1) $\psi = A \exp(K_i, x_i)$ ------ (3)

where x_i is a four position vector as defined subsequently in four vectors. The inner product of (K_i) with four vector (x_i) gives a scalar quantity, hence (K_i) is also a four vector with following usual spatial an temporal transformations as given in derivation of Lorentz transformation relations.

$$\therefore \vec{K}' = \vec{K} + \frac{(\vec{K}\cdot\vec{v})\vec{v}}{v^2}(\beta - 1) - \beta\vec{v}\frac{\omega}{c^2} \quad \text{------} (4)$$

and $\omega' = \beta\{\omega - (\vec{K}\cdot\vec{v})\}$ Her $\beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ ------ (5)

Eq^n ①, ③ and ④ gives the direction of propagation in from s' and s'' while eq^n ⑤ inter-related the frequencies in s' and s''. Since

$$\vec{K} = \frac{2\pi}{\lambda}\cdot\frac{\vec{u}}{u} = 2\pi\nu\,\frac{\vec{u}}{u^2}$$

from eq^n ⑤ $\omega' = 2\pi\nu' = \beta\left[2\pi\nu - 2\pi\nu\left(\frac{\vec{u}\cdot\vec{v}}{u^2}\right)\right]$

$$\Rightarrow \nu' = \beta\nu\left[1 - \frac{(\vec{u}\cdot\vec{v})}{u^2}\right] \quad \text{......} \quad ⑥$$

This gives the Relativistic Doppler's effect and is different from classical case due to the factor

$$\beta = \frac{1}{\sqrt{1 - V^2/c^2}}$$

If we apply it to the case of light propagating in free space, we $u = c$ and for the source system attached to the frame s'' we get $V' = V_0$

Hence from eq^n ⑥

$$\therefore \cancel{T_0 = t_2 - t_1} \quad \nu_0 = \nu\beta\left[1 - \frac{\vec{c}\cdot\vec{v}}{c^2}\right]$$

$$\text{or,}\quad \nu = \frac{\nu_0\sqrt{1 - \frac{v^2}{c^2}}}{\left[1 - \frac{\vec{c}\cdot\vec{v}}{c^2}\right]} \quad \text{......} \quad ⑦$$

In non relativistic case where $\frac{v^2}{c^2} << 1$, we get

$$\nu = \frac{\nu_0}{\left[1 - \frac{\vec{c}\cdot\vec{v}}{c^2}\right]} \quad \text{......} \quad ⑧$$

This is the usual classical relation. The following cases of interest may be studied with the help of eq^n ⑦

(i) In case the source of light moves radially away from the observer, we have $\vec{c}\cdot\vec{v} = -cv$, In this case eq^n ⑦ becomes

$$\nu = \nu_0\sqrt{\frac{c - v}{c + v}} \quad \text{......} \quad ⑨$$

which shows that $\nu_0 >> \nu$, However in classical case in which $\frac{v^2}{c^2} << 1$, we get from (7)

$$\nu = \nu_0 \left[\frac{c}{c+v} \right] \quad \text{------} \quad (10)$$

(ii) In alternate case in which the source moves radially towards the observer, we have

$\vec{c} \cdot \vec{v} = cv$ and from eqⁿ (7)

$$\nu = \nu_0 \sqrt{\frac{c+v}{c-v}} \quad \text{------} \quad (11)$$

which shows that $\nu > \nu_0$ in such a case, However in classical case when $\frac{v^2}{c^2} << 1$

from eqⁿ (7) $\quad \nu = \nu_0 \left[\frac{c}{c-v} \right] \quad \text{------} \quad (12)$

(iii) In case when the source moves at right angle to the observer, $\vec{c} \cdot \vec{v} = 0$.

from eqⁿ (7)

$$\nu = \nu_0 \sqrt{1 - \frac{v^2}{c^2}} \quad \text{------} \quad (13)$$

This shows that no transverse Doppler effect exists in classical case while it exists in relativistic case only with $\nu_0 > \nu$.